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# The genus distributions of directed antiladders in orientable surfaces☆

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**Abstract**

Although there are some results concerning genus distributions of graphs, little is known about those of digraphs. In this work, the genus distributions of 4-regular directed antiladders in orientable surfaces are obtained.


*Keywords:* Graph; Embedding; Genus; Genus polynomial; Surfaces

## 1. Introduction

Thomassen proved that computing a genus is NP-complete for general graphs [11]. Some papers have given the embedding distributions of graphs, such as [3–7,9,10]. In this work, we consider cellular embeddings of loopless digraphs into connected orientable surfaces. Research in this area has been reported in [1,2] etc. We consider only Eulerian digraphs with $\text{indeg}(x) = \text{outdeg}(x) \geq 2$, for each vertex $x$. By an *embedding* of a digraph $D$ into an orientable surface $S$, we mean that the arcs and vertices of $D$ are placed on the surface $S$, with arcs meeting only at mutually incident vertices in such a way that the orientation of a region is consistent with that of the arcs which make up its boundary. The *regions* of an embedding are the components of the complement of the digraph in an orientable surface. A *cellular embedding* is an embedding such that each region is homeomorphic to an open disk. Two embeddings $f : D \to S$ and $g : D \to S$ are *the same* if there is an orientation preserving homeomorphism $h : S \to S$ such that $hf = g$.

The *genus polynomial* of a digraph $D$ is defined as follows:

$f_D(x) = \sum_{i=0}^{\infty} g_i(D)x^i$, where $g_i(D)$ is the number of different embeddings of $D$ into the oriented surface of genus $i$.

Although there are some results concerning genus distributions of graphs, little is known about digraphs. In this work, on the basis of the joint tree method introduced by Liu [8], the genus distributions of new kinds of digraphs $DL_n$ are obtained.

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An orientable surface $S$ can be represented as a cyclic sequence, i.e., a string of letters, where each letter appears exactly twice and the two occurrences of each letter have distinct upper indices "+" (which is always omitted) and "−". This is called an *algebraic representation* of $S$. (Refer to [8,9] for details.)

It can be seen that the algebraic representation of each surface is equivalent to only one of the following canonical forms:

$$O_i = \begin{cases} (a_0 a_0^-), & \text{if the surface is a sphere} \\ \left( \prod_{k=1}^{i} a_k b_k a_k^- b_k^- \right), & \text{if the genus of a surface is } i, \end{cases}$$

i.e., $O_i$ is the *canonical representation* of a surface with genus $i$. The genus of a surface is also called *the genus of its algebraic representation* which is denoted by $o(O_i) = i$.

Let $D$ be a digraph and $T$ be a spanning tree of $D$. For a given embedding of $D$ and each non-tree arc $e$, $e$ splits into two semiarcs $e^+$ and $e^-$. It is obvious that the digraphs obtained by splitting each of cotree arcs is a tree which is called a *joint tree* $\widetilde{T}$. (We shall use $e$ for $e^+$ for the sake of convenience without confusion in this work.)

For a joint tree $\widetilde{T}$ of $D$, a cyclic sequence of all letters of semiarcs along the clockwise (or anticlockwise) rotation is an algebraic representation of the embedded surface of $D$.

It is known that the genus distribution is independent of the choice of a tree.

Let $A$ and $B$ be sections of successive letters in cyclic orders (called *linear sequences*) and $\emptyset$ be an empty set. Let $e$, $e_1$ and $e_2$ be distinct letters. By an *elementary transformation*, we mean one of the following three operations and their inverses on an algebraic representation $S$ of surfaces:

OP1 $S = (AB) \Rightarrow S = (Aee^-B)$, where $AB \neq \emptyset$ and $e \neq AB$;
OP2 $S = (Ae_1e_2Be_2^-e_1^-) \Rightarrow S = (AeBe^-)$;
OP3 $S = (AaBCa^-D) \Rightarrow S = (BaADa^-C)$.

If two algebraic representations $S_1$ and $S_2$ can be converted from one to another by finite sequences of elementary transformations, then we say that $S_1$ and $S_2$ are *equivalent*, denoted by $S_1 \sim S_2$.

It was shown that the relation $\sim$ is indeed an equivalence relation and the genera of surfaces are invariant under the elementary transformations.

In what follows, the parentheses in each algebraic representation are always omitted for the sake of brevity. The following Lemmas 1 and 2 have been proved in [8,9].

**Lemma 1.** *Let $A, B, C, D$ and $E$ be linear sequences, where $x \neq y \neq z$ and $x, y, z, x^-, y^-, z^- \neq ABCDE$. Then:*

(1) $AxByCx^-Dy^-E \sim ADCBExyx^-y^-$.
(2) $xABx^-CD \sim xBAx^-CD \sim xABx^-DC$.
(3) $AxBx^-yCy^-zDz^- \sim xBx^-AyCy^-zDz^- \sim xBx^-yCy^-AzDz^- \sim AxBx^-yCy^-zDz^- \sim BxAx^-yCy^-zDz^- \sim CxAx^-yBy^-zDz^- \sim DxAx^-yBy^-zCz^-$.

**Lemma 2.** *Let $S$ and $S'$ be two algebraic representations of surfaces, if $S \sim S'xyx^-y^-$, and $x, y, x^-, y^- \notin S'$, then* $o(S) = o(S') + 1$.

## 2. Main results

Let $C$ be a di-circuit with $4n + 4$ vertices, say $u, u_1, u_2, \ldots, u_{2n-1}, u_{2n}, u', v, v_1, v_2, \ldots, v_{2n-1}, v_{2n}, v'$ along the orientation of $C$. An *antiladder* denoted by $DL_n$ is a digraph which is obtained from $C$ by adding $2n + 1$ pairs of diagonal arcs $a_i^1 = \langle u_i, v_{i+1}\rangle, a_i^2 = \langle v_{i+1}, u_i\rangle, b_i^1 = \langle u_{i+1}, v_i\rangle, b_i^2 = \langle v_i, u_{i+1}\rangle$ for each odd $i$ and arcs $b = \langle u, v\rangle, c = \langle v, u\rangle$. $DL_n$ is a non-planar digraph for $n \geq 2$. (See Fig. 1.)

Let $C - a$ be a spanning tree, where $a = \langle v', u\rangle$. Let the joint tree $\tilde{T}$ have a anticlockwise rotation at each vertex. (See Fig. 2.)

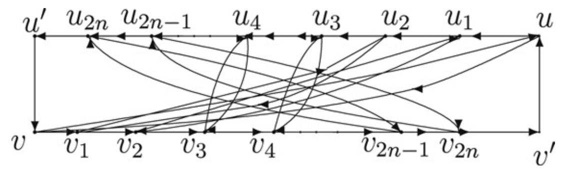


Fig. 1. $DL_n$.

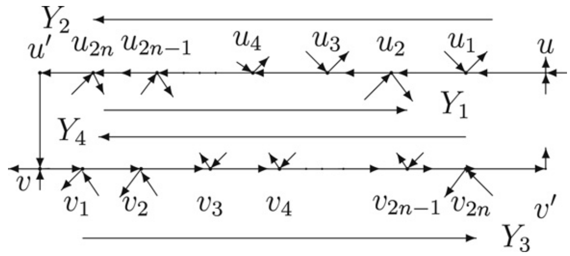


Fig. 2. A spanning tree.

Let $Y_1 = y_{k_1}y_{k_2}y_{k_3}\cdots y_{k_r}$, $Y_2 = y_{k_{r+1}}y_{k_{r+2}}y_{k_{r+3}}\cdots y_{k_{2n}}$, $Y_3 = y^-_{m_1}y^-_{m_2}y^-_{m_3}\cdots y^-_{m_s}$, and $Y_4 = y^-_{m_{s+1}}y^-_{m_{s+2}}y^-_{m_{s+3}}\cdots y^-_{m_{2n}}$, where

$$y_{i_j} = \begin{cases} a^2_{i_j}a^1_{i_j}, & k_r \le i_j \le k_1 \text{ and } i_j \equiv 1 \pmod 2 \\ b^2_{i_j-1}b^1_{i_j-1}, & k_r \le i_j \le k_1 \text{ and } i_j \equiv 0 \pmod 2 \\ a^1_{i_j}a^2_{i_j}, & k_{r+1} \le i_j \le k_{2n} \text{ and } i_j \equiv 1 \pmod 2 \\ b^1_{i_j-1}b^2_{i_j-1}, & k_{r+1} \le i_j \le k_{2n} \text{ and } i_j \equiv 0 \pmod 2 \end{cases}$$

$$y^-_{i_j} = \begin{cases} b^{2-}_{i_j}b^{1-}_{i_j}, & m_1 \le i_j \le m_s \text{ and } i_j \equiv 1 \pmod 2 \\ a^{2-}_{i_j-1}a^{1-}_{i_j-1}, & m_1 \le i_j \le m_s \text{ and } i_j \equiv 0 \pmod 2 \\ b^{1-}_{i_j}b^{2-}_{i_j}, & m_{2n} \le i_j \le k_{s+1} \text{ and } i_j \equiv 1 \pmod 2 \\ a^{1-}_{i_j-1}a^{2-}_{i_j-1}, & m_{2n} \le i_j \le k_{s+1} \text{ and } i_j \equiv 0 \pmod 2. \end{cases}$$

Let $M^n_1 = aY_1Y_4a^-Y_2Y_3$, $M^n_2 = Y_1Y_4aY_2a^-bY_3b^-$, $M^n_3 = Y_1aY_2a^-bY_3b^-cY_4c^-$ and $M^n_4 = Y_2Y_3aY_1a^-bY_4b^-$. Let $R^n_i$ denote the set of surfaces whose algebraic representations have the forms of $M^n_i$ for each $(1 \le i \le 4)$. In fact, each algebraic representation of surfaces in $R^n_i$ has at least $8n + 2$ letters of semiarcs.

**Theorem.** *Let $g_i(DL_n)$ be the number of different embeddings of the directed antiladder $DL_n$ into orientable surfaces of genus $i$, $g_{i_j}(n)$ be the number of surfaces with genus $i$ in $R^n_j$ for $n \ge 1, i \ge 0$, $r_{i_j} = g_{i_j}(n-1)$. Then,*

$$g_i(DL_n) = 2g_{(i-1)_1}(n) + 2g_{i_2}(n), \qquad \textit{where}$$
$$g_{i_j}(n) = \begin{cases} 4r_{(i-2)_1} + 8r_{(i-1)_2} + 2r_{i_2} + 2r_{i_3}, & \textit{if } j = 1 \textit{ and } 0 \le i \le 2n; \\ 4r_{(i-2)_2} + r_{(i-1)_1} + 4r_{(i-2)_1} + r_{i_3} + 4r_{(i-1)_3} + 2r_{(i-1)_2}, & \textit{if } j = 2 \textit{ and } 0 \le i \le 2n+1; \\ 4r_{(i-2)_3} + 2r_{(i-1)_2} + 8r_{(i-2)_2} + 2r_{(i-2)_1}, & \textit{if } j = 3 \textit{ and } 0 \le i \le 2n+1. \end{cases}$$

**Proof.** The cotree semiarcs $b$ and $c$ in $DL_n$ are either both on one side or both on the other side of tree $T$ to preserve alternation of the in-arc and out-arc; and similarly for $b^-$ and $c^-$. Thus the joint tree $\tilde{T}$ has four types as follows:

(1) $= Y_1abcY_2c^-b^-Y_3a^-Y_4$; (2) $= Y_1abcY_2Y_3a^-Y_4b^-c^-$; (3) $= Y_1cbaY_2c^-b^-Y_3a^-Y_4$; (4) $= Y_1cbaY_2Y_3a^-Y_4b^-c^-$. By OPs2–3 and Lemmas 1 and 2, we have (1) $\sim Y_1adY_2d^-Y_3a^-Y_4 \sim dY_2d^-Y_4Y_1aY_3a^- \sim M^n_2$. Thus the number of different embedded surfaces with genus $i$ whose algebraic representations are in the form (1) is $g_{i_2}(n)$. Similarly, it can be checked that the number of different embedded surfaces of genus $i$ whose algebraic

representations are in the form (2) is equal to those in the form (3), so (2) and (3) contribute $2g_{(i-1)_1(n)}$ to $g_i(DL_n)$. (4) $\sim Y_2Y_3a^-Y_4ad^-Y_1d \sim M_4^n$. From the structure of $Y_i$, it can be seen that there exists a bijection $\phi$ from $R_4^n$ to $R_2^n$ such that $\phi(M_4^n) = M_2^n$, so type (4) contributes $g_{i_2}(n)$ to $g_i(DL_n)$. As a consequence: $g_i(DL_n) = 2g_{i_2}(n) + 2g_{(i-1)_1(n)}$ holds.

In what follows, we consider $g_{i_j}(n)$. Since the proof is similar for each $j$, we just prove $j = 2$.

$DL_n$ is obtained from $DL_{n-1}$ by adding four vertices and four arcs, denoted as $d, e, p$ and $q$. For any surface $S$ in $R_2^{n-1}$, let $N_2^{n-1}$ be an algebraic representation of $S$; thus $N_2^{n-1}$ can be written as $N_2^{n-1} = X_1X_4aX_2a^-bX_3b^-$, where $X_i$ are letters such that $N_2^{n-1}$ have the same form as $N_2^n$, but the number of letters of semiarcs in $N_2^{n-1}$ is 8 less than that of $N_2^n$. From a relation of the joint tree of $DL_n$ and that of $DL_{n-1}$, we know that $N_2^n$ surfaces in $R_2^n$ can be obtained from $N_2^{n-1}$ through the following 16 cases:

(5) $X_1d^-e^-p^-q^-X_4aX_2depqa^-bX_3b^-$; (6) $X_1X_4aX_2depqa^-bX_3q^-p^-e^-d^-b^-$;
(7) $X_1p^-q^-X_4aX_2depqa^-bX_3e^-d^-b^-$; (8) $X_1d^-e^-X_4aX_2depqa^-bX_3q^-p^-b^-$;
(9) $qpedX_1d^-e^-p^-q^-X_4aX_2a^-bX_3b^-$; (10) $qpedX_1X_4aX_2a^-bX_3q^-p^-e^-d^-b^-$;
(11) $qpedX_1p^-q^-X_4aX_2a^-bX_3e^-d^-b^-$; (12) $qpedX_1d^-e^-X_4aX_2a^-bX_3q^-p^-b^-$;
(13) $edX_1d^-e^-p^-q^-X_4aX_2pqa^-bX_3b^-$; (14) $edX_1X_4aX_2pqa^-bX_3q^-p^-e^-d^-b^-$;
(15) $edX_1p^-q^-X_4aX_2pqa^-bX_3e^-d^-b^-$; (16) $edX_1d^-e^-X_4aX_2pqa^-bX_3q^-p^-b^-$;
(17) $qpX_1d^-e^-p^-q^-X_4aX_2dea^-bX_3b^-$; (18) $qpX_1X_4aX_2dea^-bX_3q^-p^-e^-d^-b^-$;
(19) $qpX_1p^-q^-X_4aX_2dea^-bX_3e^-d^-b^-$; (20) $qpX_1d^-e^-X_4aX_2dea^-bX_3q^-p^-b^-$.

After checking from OPs1–3 and Lemmas 1 and 2, we can see that: Cases (5), (10), (15), (20) contribute $4r_{(i-2)_2}$ to $g_{i_2}(n)$; Case (6) contributes $r_{(i-1)_1}$ to $g_{i_2}(n)$; Cases (7), (8), (14), (18) contribute $4r_{(i-2)_1}$ to $g_{i_2}(n)$; Case (9) contributes $r_{i_3}$ to $g_{i_2}(n)$; Cases (11), (12), (13), (17) contribute $4r_{(i-1)_3}$ to $g_{i_2}(n)$; Cases (16), (19) contribute $2r_{(i-1)_2}(n-1)$ to $g_{i_2}(n)$; thus the theorem is proved. $\square$

Let $g_{0_j}(0) = 1$, $g_{i_j}(0) = 0$ for all $i \neq 0$ and $g_{i_j}(n) = 0$ for all $i < 0$; then by applying the theorem, the genus polynomial of $DL_n$ for a given $n$ can be obtained. The genus polynomials $f_{DL_n}(x)$ for $n = 0, 1, 2, \ldots, 4$ are calculated as follows:

$$f_{DL_0}(x) = 2 + 2x;$$
$$f_{DL_1}(x) = 2 + 22x + 32x^2 + 8x^3;$$
$$f_{DL_2}(x) = 20x + 180x^2 + 504x^3 + 288x^4 + 32x^5;$$
$$f_{DL_3}(x) = 8x + 216x^2 + 1712x^3 + 5360x^4 + 6912x^5 + 2048x^6 + 128x^7;$$
$$f_{DL_4}(x) = 112x^2 + 2224x^3 + 16\,576x^4 + 58\,784x^5 + 99\,200x^6 + 71\,936x^7 + 12\,800x^8 + 512x^9.$$

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